Integral Generalized Binomial Coefficients of Multiplicative Functions

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Definitions

1. Generalized Binomial Coefficients:

$$\binom{n}{m}_f = \frac{\prod_{i=1}^{n} f(i)}{\prod_{i=1}^{m} f(i) \prod_{i=1}^{n-m} f(i)}$$

2. Multiplicative Functions:

$f(ab) = f(a)f(b)$ when a and b are relatively prime

$f(1) = 1$

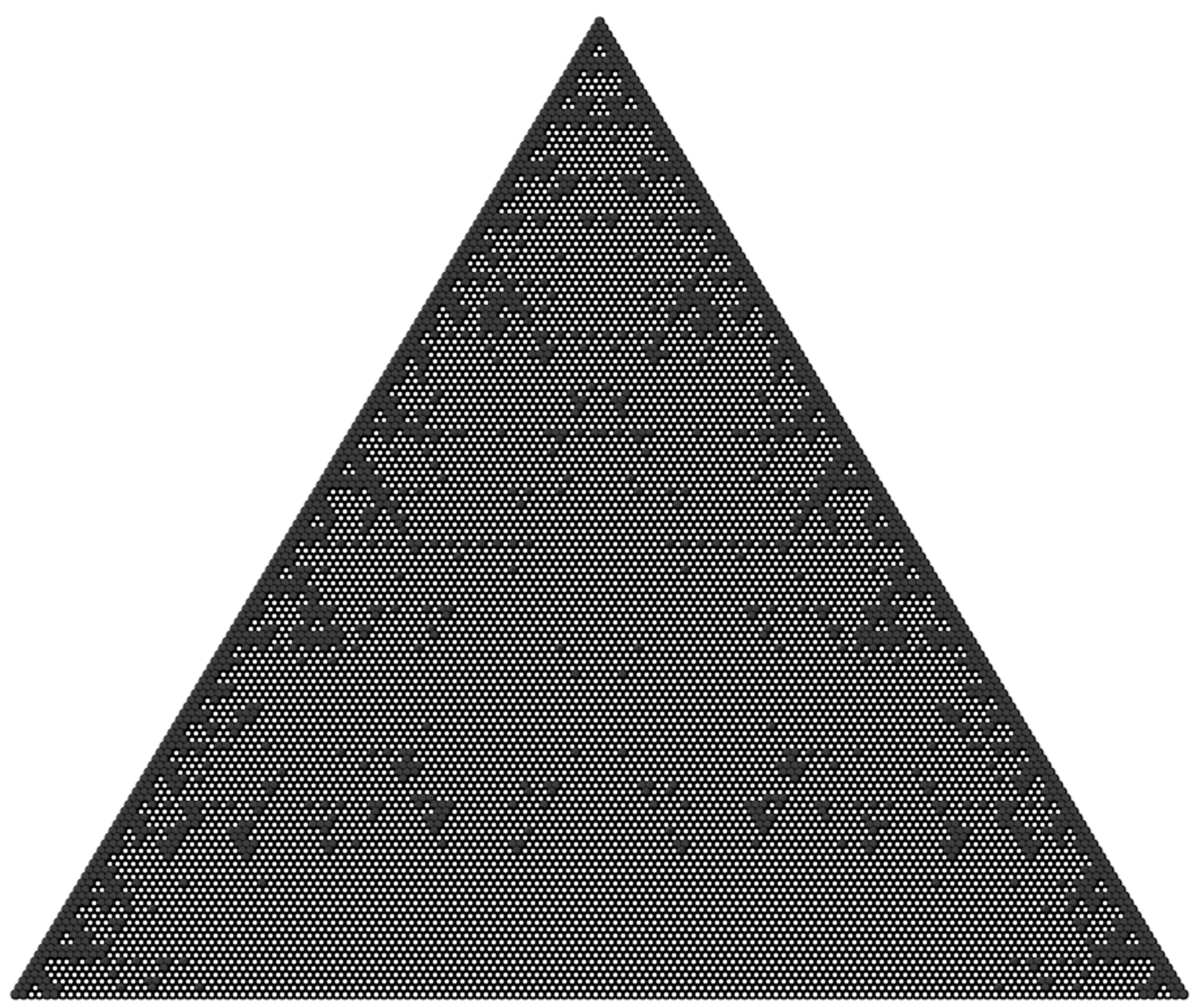
Goal

Determine a pattern for when the generalized binomial coefficeints of multiplicative functions are integral

Background

The following corollary was developed by Tom Edgar and Michael Spivey which determines when the generalized binomial coefficeints are integral for all multiplicative functions:

Corollary 8. *Let n and m be nonnegative integers. Then $\binom{n}{m}_f$ is an integer for all multiplicative functions $f : \mathbb{N} \mapsto \mathbb{N}$ if and only if for all $p \leq n$ there exists an $s_p \geq 0$ such that $\epsilon_i^p = 1$ for all $i < s_p$ and $\epsilon_i^p = 0$ for all $i \geq s_p$.*



This triangle is laid out like Pascal's Triangle. Black dots represent when $\binom{n}{m}_f$ is always integral for any multiplicative function f.

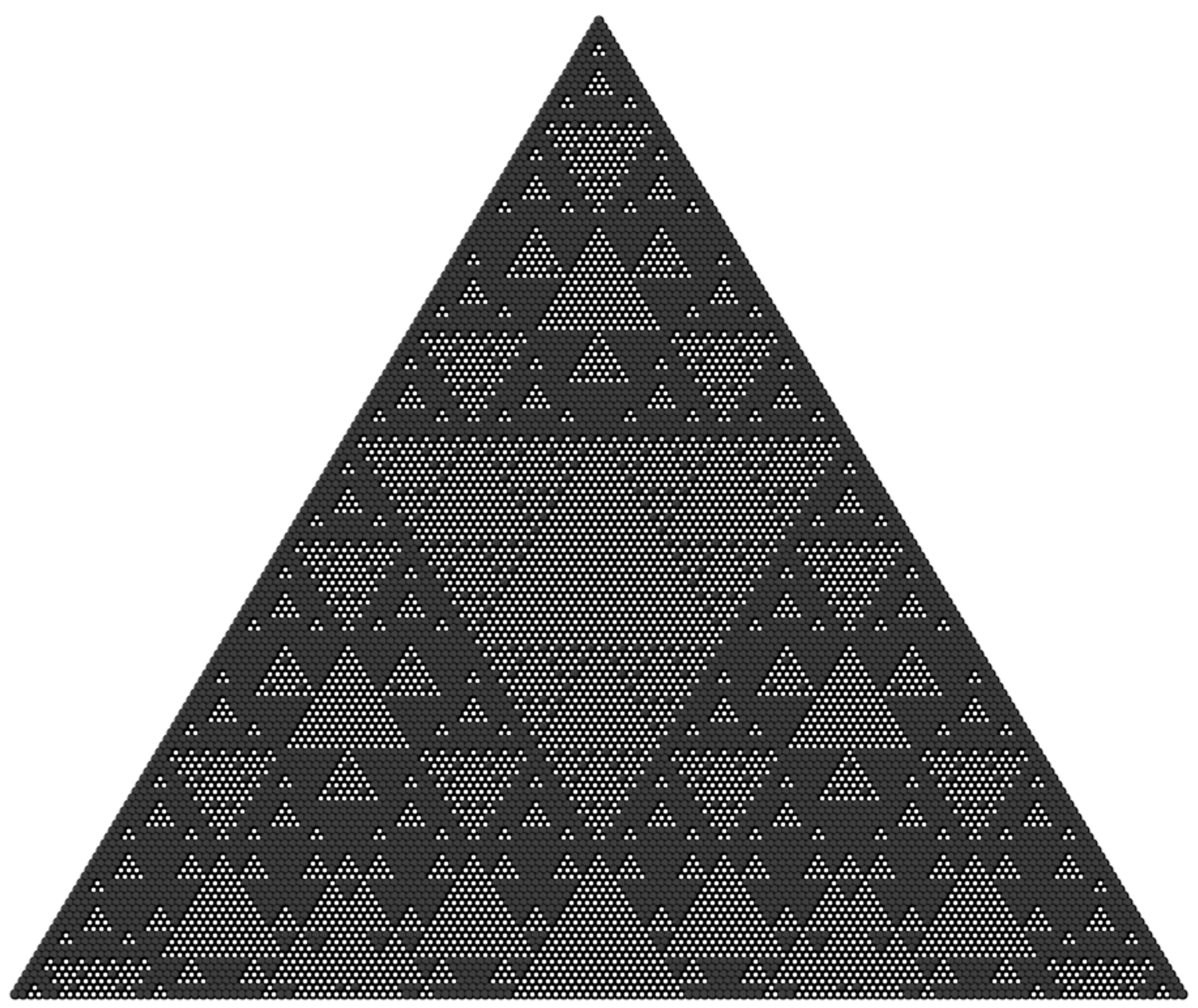
Results

- $\binom{n}{k}_f$ has a repeated pattern that cycles after $\prod_{p \leq k} p^{d_p(k)}$ where $d_p(k)$ is the number of digits contained in the base p representation of k.
- The central binomial coefficients $\binom{2n}{n}_f$ are integral only for n = 0,1,3 for $n \leq 2^{1000000}$.
- $\binom{pn}{n}_f$ is not integral when n is a multiple of p for prime p.

Multiplicative but Non-divisible Functions

1. Ruler Function $r(n)$

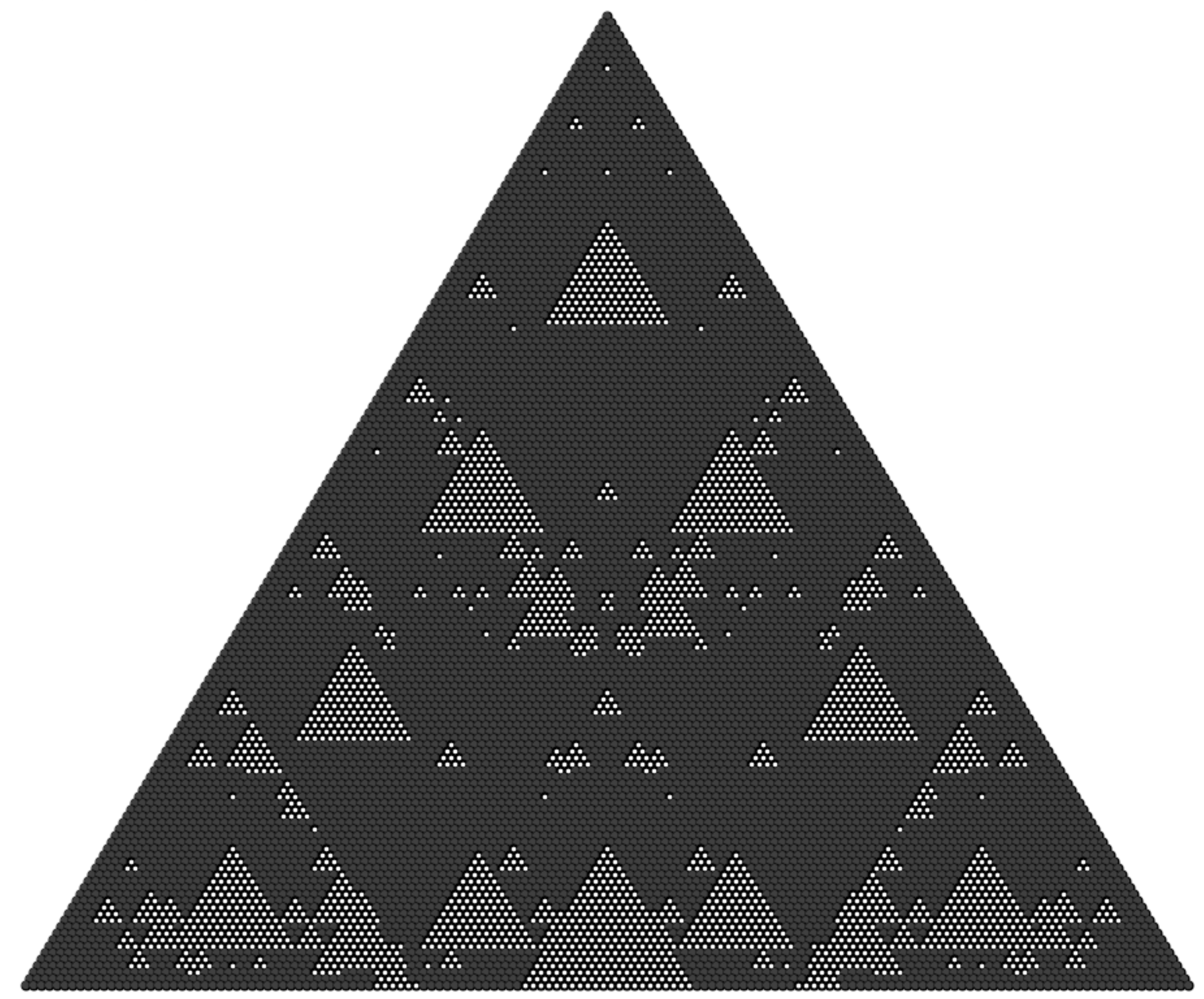
Defined as the largest power of 2 that divides $2n$. For example, $r(8) = 4$ because the largest power of 2 that divides 16 is 2^4.



- $\binom{n}{2}_r$ and $\binom{n+1}{2}_r$ is integral iff n is of the form $2^{2j+1}(1+2i)$.
- $\binom{n}{3}_r$ is not integral iff n is of the form $4^j(1+2i)$.
- $\binom{n}{m}_r$ is integral when $n = 2k-2$ and $n = 2k-1$.

2. Divisor Function $\tau(n)$

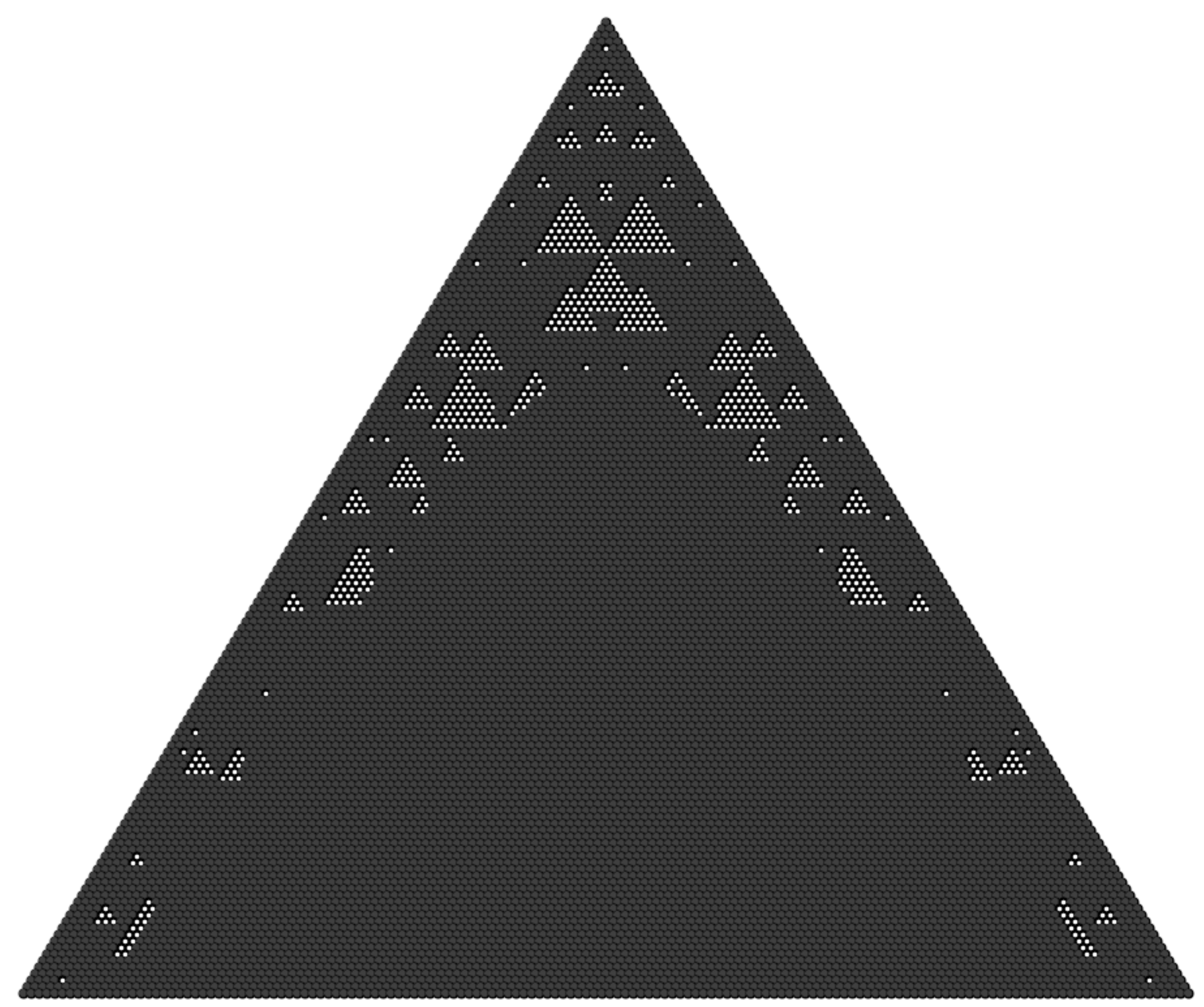
Defined as the number of divisors less than or equal to n. For example, $\tau(10) = 4$ because 10 has divisors $1, 2, 5$, and 10.



- The first four columns are entirely integral.

3. Sum of Divisors Function $\sigma(n)$

Defined as the sum of all divisors less than or equal to n. For example, $\sigma(10) = 18$ because $1+2+5+10 = 18$.



References

1. Edgar, Tom and Michael Z. Spivey. Multiplicative Functions, Generalized Binomial Coefficients, and Generalized Catalan Numbers. Preprint.
2. Erdos, P. and R. L. Graham. Old and New Problems and Results in Combinatorial Number Theory. L'Enseignement Mathmatique Universit de Genve, Vol. 28, 1980.
3. OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequence, http://oeis.org